

EULER SEQUENCE AND KOSZUL COMPLEX OF A MODULE

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ABSTRACT. We construct relative and global Euler sequences of a module. We apply it to prove some acyclicity results of the Koszul complex of a module and to compute the cohomology of the sheaves of (relative and absolute) differential p -forms of a projective bundle. In particular we generalize Bott's formula for the projective space to a projective bundle over a scheme of characteristic zero.

INTRODUCTION

This paper deals with two related questions: the acyclicity of the Koszul complex of a module and the cohomology of the sheaves of (relative and absolute) differential p -forms of a projective bundle over a scheme.

Let M be a module over a commutative ring A . One has the Koszul complex $\text{Kos}(M) = \Lambda^* M \otimes_A S^* M$, where $\Lambda^* M$ and $S^* M$ stand for the exterior and symmetric algebras of M . It is a graded complex $\text{Kos}(M) = \bigoplus_{n \geq 0} \text{Kos}(M)_n$, whose n -th graded component $\text{Kos}(M)_n$ is the complex:

$$0 \rightarrow \Lambda^n M \rightarrow \Lambda^{n-1} M \otimes M \rightarrow \Lambda^{n-2} M \otimes S^2 M \rightarrow \cdots \rightarrow M \otimes S^{n-1} M \rightarrow S^n M \rightarrow 0$$

It has been known for many years that $\text{Kos}(M)_n$ is acyclic for $n > 0$, provided that M is a flat A -module or n is invertible in A (see [3] or [10]). It was conjectured in [11] that $\text{Kos}(M)$ is always acyclic. A counterexample in characteristic 2 was given in [5], but it is also proved there that $H_\mu(\text{Kos}(M)_\mu) = 0$ for any M , where μ is the minimal number of generators of M . Leaving aside the case of characteristic 2 (whose pathology is clear for the exterior algebra), we prove two new evidences for the validity of the conjecture (for A noetherian): firstly, we prove (Theorem 1.6) that, for any finitely generated M , $\text{Kos}(M)_n$ is acyclic for $n \gg 0$; secondly, we prove (Theorem 1.7) that if I is an ideal locally generated by a regular sequence, then $\text{Kos}(I)_n$ is acyclic for any $n > 0$. These two results are a consequence of relating the Koszul complex $\text{Kos}(M)$ with the geometry of the space $\mathbb{P} = \text{Proj } S^* M$, as follows:

First of all, we shall reformulate the Koszul complex in terms of differential forms of $S^* M$ over A : the canonical isomorphism $\Omega_{S^* M/A} = M \otimes_A S^* M$ allows us to interpret the Koszul complex $\text{Kos}(M)$ as the complex of differential forms $\Omega_{S^* M/A}^*$ whose differential, $i_D: \Omega_{S^* M/A}^p \rightarrow \Omega_{S^* M/A}^{p-1}$, is the inner product with the A -derivation $D: S^* M \rightarrow S^* M$ consisting in multiplication by n on $S^n M$. By homogeneous localization, one obtains a complex of $\mathcal{O}_{\mathbb{P}}$ -modules $\widetilde{\text{Kos}}(M)$ on \mathbb{P} .

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Our first result (Theorem 1.4) is that the complex $\widetilde{\text{Kos}}(M)$ is acyclic with factors (cycles or boundaries) the sheaves $\Omega_{\mathbb{P}/A}^p$. Moreover, one has a natural morphism

$$\text{Kos}(M)_n \rightarrow \pi_*[\widetilde{\text{Kos}}(M) \otimes \mathcal{O}_{\mathbb{P}}(n)]$$

with $\pi: \mathbb{P} \rightarrow \text{Spec } A$ the canonical morphism. In Theorem 1.5 we give (cohomological) sufficient conditions for the acyclicity of the complexes $\text{Kos}(M)_n$ and $\pi_*[\widetilde{\text{Kos}}(M) \otimes \mathcal{O}_{\mathbb{P}}(n)]$. These conditions, under noetherian hypothesis, are satisfied for $n \gg 0$, thus obtaining Theorem 1.6. The acyclicity of the Koszul complex of a locally regular ideal follows then from Theorem 1.5 and the theorem of formal functions.

The advantage of expressing the Koszul complex $\text{Kos}(M)$ as $(\Omega_{S^*M/A}^\bullet, i_D)$ is two-fold. Firstly, it makes clear its relationship with the De Rham complex $(\Omega_{S^*M/A}^\bullet, d)$: The Koszul and De Rham differentials are related by Cartan's formula: $i_D \circ d + d \circ i_D = \text{multiplication by } n \text{ on } \text{Kos}(M)_n$. This yields a splitting result (Proposition 1.10 or Corollary 1.11) which will be essential for some cohomological results in section 3 as we shall explain later on. Secondly, it allows a natural generalization (which is the subject of section 2): If A is a k -algebra, we define the complex $\text{Kos}(M/k)$ as the complex of differential forms (over k), $\Omega_{S^*M/k}^\bullet$ whose differential is the inner product with the same D as before. Again, one has that $\text{Kos}(M/k) = \bigoplus_{n \geq 0} \text{Kos}(M/k)_n$ and it induces, by homogeneous localization, a complex $\widetilde{\text{Kos}}(M/k)$ of modules on \mathbb{P} which is also acyclic and whose factors are the sheaves $\Omega_{\mathbb{P}/k}^p$ (Theorem 2.1). We can reproduce the aforementioned results about the complexes $\text{Kos}(M)_n$, $\widetilde{\text{Kos}}(M)$, for the complexes $\text{Kos}(M/k)_n$, $\widetilde{\text{Kos}}(M/k)$.

Section 3 deals with the second subject of the paper: let \mathcal{E} be a locally free module of rank $r + 1$ on a k -scheme X and let $\pi: \mathbb{P} \rightarrow X$ be the associated projective bundle, i.e., $\mathbb{P} = \text{Proj } S^*\mathcal{E}$. There are well known results about the (global and relative) cohomology of the sheaves $\Omega_{\mathbb{P}/X}^p(n)$ and $\Omega_{\mathbb{P}/k}^p(n)$ (we are using the standard abbreviated notation $\mathcal{N}(n) = \mathcal{N} \otimes \mathcal{O}_{\mathbb{P}}(n)$) due to Deligne, Verdier and Berthelot-Illusie ([4], [12], [1]) and about the cohomology of the sheaves $\Omega_{\mathbb{P}^r}^p(n)$ of the ordinary projective space due to Bott (the so called Bott's formula, [2]). We shall not use their results; instead, we reprove them and we obtain some new results, overall when X is a \mathbb{Q} -scheme. Let us be more precise:

In Theorem 3.4 we compute the relative cohomology sheaves $R^i \pi_* \Omega_{\mathbb{P}/X}^p(n)$, obtaining Deligne's result (see [4] and also [12]) and a new (splitting) result, in the case of a \mathbb{Q} -scheme, concerning the sheaves $\pi_* \Omega_{\mathbb{P}/X}^p(n)$ and $R^r \pi_* \Omega_{\mathbb{P}/X}^p(-n)$ for $n > 0$. We obtain Bott formula for the projective space as a consequence. In Theorem 3.11 we compute the relative cohomology sheaves $R^i \pi_* \Omega_{\mathbb{P}/k}^p(n)$, obtaining Verdier's results (see [12]) and improving them in two ways: first, we give a more explicit description of $\pi_* \Omega_{\mathbb{P}/k}^p(n)$ and of $R^r \pi_* \Omega_{\mathbb{P}/k}^p(-n)$ for $n > 0$; secondly, we obtain a splitting result for these sheaves when X is a \mathbb{Q} -scheme (as in the relative case).

Regarding Bott's formula, we are able to generalize it for a projective bundle, computing the dimension of the cohomology vector spaces $H^q(\mathbb{P}, \Omega_{\mathbb{P}/X}^p(n))$ and $H^q(\mathbb{P}, \Omega_{\mathbb{P}/k}^p(n))$ when X is a proper k -scheme of characteristic zero (Corollaries 3.7 and 3.14).

It should be mentioned that these results make use of the complexes $\widetilde{\text{Kos}}(\mathcal{E})$ (as Deligne and Verdier) and $\widetilde{\text{Kos}}(\mathcal{E}/k)$. The complex $\widetilde{\text{Kos}}(\mathcal{E})$ is essentially equivalent

to the exact sequence

$$0 \rightarrow \Omega_{\mathbb{P}/X} \rightarrow (\pi^* \mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}}(-1) \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow 0$$

which is usually called Euler sequence. The complex $\widetilde{\text{Kos}}(\mathcal{E}/k)$ is equivalent to the exact sequence

$$0 \rightarrow \Omega_{\mathbb{P}/k} \rightarrow \widetilde{\Omega}_{B/k} \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow 0$$

with $B = S^* \mathcal{E}$, which we have called global Euler sequence. These sequences still hold for any A -module M (which we have called relative and global Euler sequences of M). The aforementioned results about the acyclicity of the Koszul complexes of a module obtained in sections 1 and 2 are a consequence of this fact.

1. RELATIVE EULER SEQUENCE OF A MODULE AND KOSZUL COMPLEXES

Let (X, \mathcal{O}) be a scheme and let \mathcal{M} be quasi-coherent \mathcal{O} -module. Let $\mathcal{B} = S^* \mathcal{M}$ be the symmetric algebra of \mathcal{M} (over \mathcal{O}), which is a graded \mathcal{O} -algebra: the homogeneous component of degree n is $\mathcal{B}_n = S^n \mathcal{M}$. The module $\Omega_{\mathcal{B}/\mathcal{O}}$ of Khaler differentials is a graded \mathcal{B} -module in a natural way: $\mathcal{B} \otimes_{\mathcal{O}} \mathcal{B}$ is a graded \mathcal{O} -algebra, with $(\mathcal{B} \otimes_{\mathcal{O}} \mathcal{B})_n = \bigoplus_{p+q=n} \mathcal{B}_p \otimes_{\mathcal{O}} \mathcal{B}_q$ and the natural morphism $\mathcal{B} \otimes_{\mathcal{O}} \mathcal{B} \rightarrow \mathcal{B}$ is a degree 0 homogeneous morphism of graded algebras. Hence, the kernel Δ is a homogeneous ideal and $\Delta/\Delta^2 = \Omega_{\mathcal{B}/\mathcal{O}}$ is a graded \mathcal{B} -module. If $b_p, b_q \in \mathcal{B}$ are homogeneous of degree p, q , then $b_p \text{ d } b_q$ is an element of $\Omega_{\mathcal{B}/\mathcal{O}}$ of degree $p+q$. We shall denote by $\Omega_{\mathcal{B}/\mathcal{O}}^p = \Lambda_{\mathcal{B}}^p \Omega_{\mathcal{B}/\mathcal{O}}$, the p -th exterior power of $\Omega_{\mathcal{B}/\mathcal{O}}$, which is also a graded \mathcal{B} -module in a natural way. For each \mathcal{O} -module \mathcal{N} , $\mathcal{N} \otimes_{\mathcal{O}} \mathcal{B}$ is a graded \mathcal{B} -module with gradation: $(\mathcal{N} \otimes_{\mathcal{O}} \mathcal{B})_n = \mathcal{N} \otimes_{\mathcal{O}} \mathcal{B}_n$. Then one has the following basic result:

Theorem 1.1. *The natural morphism of graded \mathcal{B} -modules*

$$\begin{aligned} \mathcal{M} \otimes_{\mathcal{O}} \mathcal{B}[-1] &\rightarrow \Omega_{\mathcal{B}/\mathcal{O}} \\ m \otimes b &\mapsto b \text{ d } m \end{aligned}$$

is an isomorphism. Hence $\Omega_{\mathcal{B}/\mathcal{O}}^p \simeq \Lambda^p \mathcal{M} \otimes_{\mathcal{O}} \mathcal{B}[-p]$, where $\Lambda^i \mathcal{M} = \Lambda_{\mathcal{O}}^i \mathcal{M}$.

The natural morphism $\mathcal{M} \otimes_{\mathcal{O}} S^i \mathcal{M} \rightarrow S^{i+1} \mathcal{M}$ defines a degree zero homogeneous morphism of \mathcal{B} -modules $\Omega_{\mathcal{B}/\mathcal{O}} = \mathcal{M} \otimes_{\mathcal{O}} \mathcal{B}[-1] \rightarrow \mathcal{B}$ which induces an \mathcal{O} -derivation (of degree 0) $D: \mathcal{B} \rightarrow \mathcal{B}$, such that $\Omega_{\mathcal{B}/\mathcal{O}} \rightarrow \mathcal{B}$ is the inner product with D . This derivation consists in multiplication by n in degree n . It induces homogeneous morphisms of degree zero:

$$i_D: \Omega_{\mathcal{B}/\mathcal{O}}^p \rightarrow \Omega_{\mathcal{B}/\mathcal{O}}^{p-1}$$

and we obtain:

Definition 1.2. The Koszul complex, denoted by $\text{Kos}(\mathcal{M})$, is the complex:

$$\cdots \longrightarrow \Omega_{\mathcal{B}/\mathcal{O}}^p \xrightarrow{i_D} \Omega_{\mathcal{B}/\mathcal{O}}^{p-1} \xrightarrow{i_D} \cdots \xrightarrow{i_D} \Omega_{\mathcal{B}/\mathcal{O}} \xrightarrow{i_D} \mathcal{B} \longrightarrow 0 \quad (1.1)$$

Via Theorem 1.1, this complex is

$$\cdots \xrightarrow{i_D} \Lambda^p \mathcal{M} \otimes_{\mathcal{O}} \mathcal{B}[-p] \xrightarrow{i_D} \cdots \xrightarrow{i_D} \mathcal{M} \otimes_{\mathcal{O}} \mathcal{B}[-1] \xrightarrow{i_D} \mathcal{B} \rightarrow 0$$

Taking the homogeneous components of degree $n \geq 0$, we obtain a complex of \mathcal{O} -modules, which we denote by $\text{Kos}(\mathcal{M})_n$:

$$0 \longrightarrow \Lambda^n \mathcal{M} \longrightarrow \Lambda^{n-1} \mathcal{M} \otimes_{\mathcal{O}} \mathcal{M} \longrightarrow \dots \longrightarrow \mathcal{M} \otimes_{\mathcal{O}} S^{n-1} \mathcal{M} \longrightarrow S^n \mathcal{M} \longrightarrow 0$$

such that $\text{Kos}(\mathcal{M}) = \bigoplus_{n \geq 0} \text{Kos}(\mathcal{M})_n$.

Now let $\mathbb{P} = \text{Proj } \mathcal{B}$ and $\pi: \mathbb{P} \rightarrow X$ the natural morphism. We shall use the following standard notations: for each $\mathcal{O}_{\mathbb{P}}$ -module \mathcal{N} , we shall denote $\mathcal{N}(n) = \mathcal{N} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_{\mathbb{P}}(n)$ and for each graded \mathcal{B} -module N we shall denote by \tilde{N} the sheaf of $\mathcal{O}_{\mathbb{P}}$ -modules obtained by homogeneous localization. We shall use without mention the following facts: homogeneous localization commutes with exterior powers and for any quasi-coherent module \mathcal{L} on X one has $(\mathcal{L} \otimes_{\mathcal{O}} \widetilde{\mathcal{B}[r]}) = (\pi^* \mathcal{L})(r)$.

Definition 1.3. Taking homogeneous localization on the Koszul complex (1.1), we obtain a complex of $\mathcal{O}_{\mathbb{P}}$ -modules, which we denote by $\widetilde{\text{Kos}}(\mathcal{M})$:

$$\dots \longrightarrow \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^d \xrightarrow{i_D} \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^{d-1} \xrightarrow{i_D} \dots \xrightarrow{i_D} \tilde{\Omega}_{\mathcal{B}/\mathcal{O}} \xrightarrow{i_D} \mathcal{O}_{\mathbb{P}} \longrightarrow 0 \quad (1.2)$$

By Theorem 1.1, $\tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^d = (\pi^* \Lambda^d \mathcal{M})(-d)$, hence $\widetilde{\text{Kos}}(\mathcal{M})$ can be written as

$$\dots \xrightarrow{i_D} (\pi^* \Lambda^d \mathcal{M})(-d) \xrightarrow{i_D} \dots \rightarrow (\pi^* \mathcal{M})(-1) \xrightarrow{i_D} \mathcal{O}_{\mathbb{P}} \rightarrow 0$$

Theorem 1.4. *The complex $\widetilde{\text{Kos}}(\mathcal{M})$ is acyclic (that is, an exact sequence). Moreover,*

$$\Omega_{\mathbb{P}/X}^p = \text{Ker} \left(\tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^p \xrightarrow{i_D} \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^{p-1} \right).$$

Hence one has exact sequences

$$0 \rightarrow \Omega_{\mathbb{P}/X}^p \rightarrow \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^p \rightarrow \Omega_{\mathbb{P}/X}^{p-1} \rightarrow 0$$

and right and left resolutions of $\Omega_{\mathbb{P}/X}^p$:

$$\begin{aligned} 0 \rightarrow \Omega_{\mathbb{P}/X}^p \rightarrow \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^p \rightarrow \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^{p-1} \rightarrow \dots \rightarrow \tilde{\Omega}_{\mathcal{B}/\mathcal{O}} \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow 0 \\ \dots \rightarrow \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^{r+1} \rightarrow \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^r \rightarrow \dots \rightarrow \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^{p+1} \rightarrow \Omega_{\mathbb{P}/X}^p \rightarrow 0 \end{aligned}$$

In particular, for $p = 1$ the exact sequence

$$0 \rightarrow \Omega_{\mathbb{P}/X} \rightarrow \tilde{\Omega}_{\mathcal{B}/\mathcal{O}} \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow 0 \quad (1.3)$$

is called the (relative) Euler sequence.

Proof. The morphism $\tilde{\Omega}_{\mathcal{B}/\mathcal{O}} \rightarrow \mathcal{O}_{\mathbb{P}}$ is surjective, since $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{B}[-1] \rightarrow \mathcal{B}$ is surjective in positive degree. Let K be the kernel. We obtain an exact sequence

$$0 \rightarrow K \rightarrow \tilde{\Omega}_{\mathcal{B}/\mathcal{O}} \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow 0$$

Since $\mathcal{O}_{\mathbb{P}}$ is free, this sequence splits locally; then, it induces exact sequences

$$0 \rightarrow \Lambda^p K \rightarrow \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^p \rightarrow \Lambda^{p-1} K \rightarrow 0$$

Joining these exact sequences one obtains the Koszul complex $\widetilde{\text{Kos}}(\mathcal{M})$. This proves the acyclicity of $\widetilde{\text{Kos}}(\mathcal{M})$. To conclude, it suffices to prove that $K = \Omega_{\mathbb{P}/X}$.

Let us first define a morphism $\Omega_{\mathbb{P}/X} \rightarrow \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}$. Assume for simplicity that $X = \text{Spec } A$. For each $b \in \mathcal{B}$ of degree 1, let U_b the standard affine open subset of \mathbb{P}

defined by $U_b = \text{Spec}(\mathcal{B}_{(b)})$, with $\mathcal{B}_{(b)}$ the 0-degree component of \mathcal{B}_b . The natural inclusion $\mathcal{B}_{(b)} \rightarrow \mathcal{B}_b$ induces a morphism $\Omega_{\mathcal{B}_{(b)}/A} \rightarrow \Omega_{\mathcal{B}_b/A} = (\Omega_{\mathcal{B}/A})_b$ which takes values in the 0-degree component, $(\Omega_{\mathcal{B}/A})_{(b)}$. Thus one has a morphism $\Omega_{\mathcal{B}_{(b)}/A} \rightarrow (\Omega_{\mathcal{B}/A})_{(b)}$, i.e. a morphism $\Gamma(U_b, \Omega_{\mathbb{P}/X}) \rightarrow \Gamma(U_b, \tilde{\Omega}_{\mathcal{B}/A})$. One checks that these morphisms glue to a morphism $f: \Omega_{\mathbb{P}/X} \rightarrow \tilde{\Omega}_{\mathcal{B}/A}$. This morphism is injective, because the inclusion $\mathcal{B}_{(b)} \rightarrow \mathcal{B}_b$ has a retract, $c_n/b^k \mapsto c_n/b^n$, which induces a retract in the differentials. The composition $\Omega_{\mathbb{P}/X} \rightarrow \tilde{\Omega}_{\mathcal{B}/A} \rightarrow \mathcal{O}_{\mathbb{P}}$ is null, as one checks in each U_b :

$$(i_D \circ f)(d(c_k/b^k)) = i_D \left(\frac{b^k d c_k - c_k d b^k}{b^{2k}} \right) = \frac{b^k i_D d c_k - c_k i_D d b^k}{b^{2k}} = 0$$

because $i_D d c_r = r c_r$ for any element c_r of degree r . Thus, we have that $\Omega_{\mathbb{P}/X}$ is contained in the kernel of $\tilde{\Omega}_{\mathcal{B}/A} \rightarrow \mathcal{O}_{\mathbb{P}}$. To conclude, it is enough to see that the image of $\tilde{\Omega}_{\mathcal{B}/A}^2 \xrightarrow{i_D} \tilde{\Omega}_{\mathcal{B}/A}$ is contained in $\Omega_{\mathbb{P}/X}$. Again, this is a computation in each U_b ; one checks the equality

$$i_D \left(\frac{d c_p \wedge d c_q}{b^{p+q}} \right) = p \frac{c_p}{b^p} d \left(\frac{c_q}{b^q} \right) - q \frac{c_q}{b^q} d \left(\frac{c_p}{b^p} \right)$$

and the right member belongs to $\Omega_{\mathcal{B}_{(b)}/A}$. \square

For each $n \in \mathbb{Z}$, we shall denote by $\widetilde{\text{Kos}}(\mathcal{M})(n)$ the complex $\widetilde{\text{Kos}}(\mathcal{M})$ twisted by $\mathcal{O}_{\mathbb{P}}(n)$ (notice that the differential of the Koszul complex is $\mathcal{O}_{\mathbb{P}}$ -linear). The differential of the complex $\widetilde{\text{Kos}}(\mathcal{M})(n)$ is still denoted by i_D .

1.1. Acyclicity of the Koszul complex of a module.

Let us denote $\widetilde{\text{Kos}}(\mathcal{M})_n := \pi_*(\widetilde{\text{Kos}}(\mathcal{M})(n))$. The natural morphisms $[\Omega_{\mathcal{B}/\mathcal{O}}^p]_n \rightarrow \pi_*[\tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^p(n)]$ give a morphism of complexes

$$\text{Kos}(\mathcal{M})_n \rightarrow \widetilde{\text{Kos}}(\mathcal{M})_n$$

and one has:

Theorem 1.5. *Let \mathcal{M} be a finitely generated quasi-coherent module on a scheme (X, \mathcal{O}) , $\mathbb{P} = \text{Proj } S^* \mathcal{M}$ and $\pi: \mathbb{P} \rightarrow X$ the natural morphism. Let d be the minimal number of generators of \mathcal{M} (i.e., it is the greatest integer such that $\Lambda^d \mathcal{M} \neq 0$) and $n > 0$. Then:*

- (1) *If $R^j \pi_*[\tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^i(n)] = 0$ for any $j > 0$ and any $0 \leq i \leq d$, then $\widetilde{\text{Kos}}(\mathcal{M})_n$ is acyclic.*
- (2) *If (1) holds and the natural morphism $[\Omega_{\mathcal{B}/\mathcal{O}}^i]_n \rightarrow \pi_*[\tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^i(n)]$ is an isomorphism for any $0 \leq i \leq d$, then $\text{Kos}(\mathcal{M})_n$ is also acyclic.*

Proof. (1) By Theorem 1.4, the complex $\widetilde{\text{Kos}}(\mathcal{M})(n)$ is acyclic. Since the (non-zero) terms of this complex are $\tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^i(n)$, the hypothesis tells us that $\pi_*(\widetilde{\text{Kos}}(\mathcal{M}) \otimes \mathcal{O}_{\mathbb{P}}(n))$ is acyclic, that is, $\widetilde{\text{Kos}}(\mathcal{M})_n$ is acyclic.

(2) By hypothesis, $\text{Kos}(\mathcal{M})_n \rightarrow \widetilde{\text{Kos}}(\mathcal{M})_n$ is an isomorphism and then $\text{Kos}(\mathcal{M})_n$ is also acyclic. \square

Theorem 1.6. *Let X be a noetherian scheme and \mathcal{M} a coherent module on X . The Koszul complexes $\text{Kos}(\mathcal{M})_n$ and $\widetilde{\text{Kos}}(\mathcal{M})_n$ are acyclic for $n > 0$.*

Proof. Indeed, the hypothesis (1) and (2) of Theorem 1.5 hold for $n \gg 0$ (see [8, Theorem 2.2.1] and [7, Section 3.3 and Section 3.4]). \square

Theorem 1.7. *Let I be an ideal of a noetherian ring A . If I is locally generated by a regular sequence, then $\text{Kos}(I)_n$ and $\widetilde{\text{Kos}}(I)_n$ are acyclic for any $n > 0$.*

Proof. In this case $\pi: \mathbb{P} \rightarrow X = \text{Spec } A$ is the blow-up with respect to I , because $S^n I = I^n$, since I is locally a regular ideal ([9]). Let d be the minimum number of generators of I . By Theorem 1.5, it suffices to see that for any A -module M and any $0 \leq i \leq d$ one has:

$$H^j(\mathbb{P}, (\pi^* M)(n-i)) = \begin{cases} 0 & , \text{ if } j > 0 \\ M \otimes_A I^{n-i} & , \text{ if } j = 0 \end{cases}$$

This is a consequence of the Theorem of formal functions (see [8, Corollary 4.1.7]). Indeed, let us denote $Y_r = \text{Spec } A/I^r$, $E_r = \pi^{-1}(Y_r)$ and $\pi_r: E_r \rightarrow Y_r$. One has that $E_r = \text{Proj } S_{A/I^r}(I/I^{r+1})$ is a projective bundle over Y_r , because I/I^{r+1} is a locally free A/I^r -module of rank d , since I is locally regular. Hence, for any module N on Y_r and any $m > -d$ one has

$$H^j(E_r, (\pi_r^* N)(m)) = \begin{cases} 0 & , \text{ if } j > 0 \\ N \otimes_{A/I^r} I^m/I^{m+r} & , \text{ if } j = 0 \end{cases}$$

Now, by the theorem of formal functions (let $m = n - i$)

$$H^j(\mathbb{P}, (\pi^* M)(m))^\wedge = \lim_{\leftarrow r} H^j(E_r, \pi_r^*(M/I^r M)(m)) = 0, \text{ for } j > 0.$$

For $j = 0$, the natural morphism $M \otimes_A I^m \rightarrow H^0(\mathbb{P}, (\pi^* M)(m))$ is an isomorphism because it is an isomorphism after completion by I :

$$\begin{aligned} H^0(\mathbb{P}, (\pi^* M)(m))^\wedge &= \lim_{\leftarrow r} H^0(E_r, \pi_r^*(M/I^r M)(m)) \\ &= \lim_{\leftarrow r} (M/I^r M) \otimes_{A/I^r} I^m/I^{m+r} \\ &= \lim_{\leftarrow r} (M \otimes_A S^m I) \otimes_A A/I^r = (M \otimes_A I^m)^\wedge \end{aligned}$$

\square

Remark 1.8. Let d be the minimum number of generators of \mathcal{M} . Since $\widetilde{\text{Kos}}(\mathcal{M})$ is acyclic and π_* is left exact, one has that $H_d(\widetilde{\text{Kos}}(\mathcal{M})_n) = 0$ for any n . On the other hand, it is proved in [5] that $H_d(\text{Kos}(\mathcal{M})_d) = 0$. One cannot expect $\text{Kos}(\mathcal{M})_n \rightarrow \widetilde{\text{Kos}}(\mathcal{M})_n$ to be an isomorphism in general. For instance, consider $X = \text{Spec } A$ with $A = k[u, v, s_1, s_2, t_1, t_2]/I$ where k is a field and $I = (-us_1 + vt_1 + ut_2, vs_1 + us_2 - vt_2, vs_2, ut_1)$. Let $M = (Ax \oplus Ay)/A(\bar{u}x + \bar{v}y)$, where \bar{u} (resp. \bar{v}) is the class of u (resp. v) in A . Then one can prove that the map $M \rightarrow \pi_* \mathcal{O}_{\mathbb{P}}(1)$ is not injective (for details we refer to section 26.21 of The Stacks project). So that the question which arises here is whether $\text{Kos}(\mathcal{M})_n \rightarrow \widetilde{\text{Kos}}(\mathcal{M})_n$ is a quasi-isomorphism. We do not know the answer, besides the acyclicity theorems for both complexes mentioned above.

1.2. Koszul versus De Rham. The exterior differential defines morphisms

$$d: \Omega_{\mathcal{B}/\mathcal{O}}^p \rightarrow \Omega_{\mathcal{B}/\mathcal{O}}^{p+1}$$

which are \mathcal{O} -linear, but not \mathcal{B} -linear. One has then the De Rham complex:

$$\mathrm{DeRham}(\mathcal{M}) \equiv 0 \rightarrow \mathcal{B} \xrightarrow{d} \Omega_{\mathcal{B}/\mathcal{O}} \xrightarrow{d} \cdots \xrightarrow{d} \Omega_{\mathcal{B}/\mathcal{O}}^p \xrightarrow{d} \Omega_{\mathcal{B}/\mathcal{O}}^{p+1} \rightarrow \cdots$$

which can be reformulated as

$$0 \rightarrow \mathcal{B} \xrightarrow{d} \mathcal{M} \otimes_{\mathcal{O}} \mathcal{B}[-1] \rightarrow \cdots \rightarrow \Lambda^p \mathcal{M} \otimes_{\mathcal{O}} \mathcal{B}[-p] \rightarrow \Lambda^{p+1} \mathcal{M} \otimes_{\mathcal{O}} \mathcal{B}[-p-1] \rightarrow \cdots$$

Taking into account that d is homogeneous of degree 0, one has for each $n \geq 0$ a complex of \mathcal{O} -modules

$$\mathrm{DeRham}(\mathcal{M})_n \equiv 0 \rightarrow S^n \mathcal{M} \rightarrow \mathcal{M} \otimes_{\mathcal{O}} S^{n-1} \rightarrow \cdots \rightarrow \Lambda^{n-1} \mathcal{M} \otimes_{\mathcal{O}} \mathcal{M} \rightarrow \Lambda^n \mathcal{M} \rightarrow 0$$

The differentials of the Koszul and De Rham complexes are related by Cartan's formula: $i_D \circ d + d \circ i_D = \text{multiplication by } n \text{ on } \Lambda^p \mathcal{M} \otimes_{\mathcal{O}} S^{n-p} \mathcal{M}$. This immediately implies the following result:

Proposition 1.9. *If X is a scheme over \mathbb{Q} , then $\mathrm{Kos}(\mathcal{M})_n$ and $\mathrm{DeRham}(\mathcal{M})_n$ are homotopically trivial for any $n > 0$. In particular, they are acyclic.*

Now we pass to homogeneous localizations. The differential $d: \Omega_{\mathcal{B}/\mathcal{O}}^p \rightarrow \Omega_{\mathcal{B}/\mathcal{O}}^{p+1}$ is compatible with homogeneous localization, since for any $\omega_{k+n} \in \Omega_{\mathcal{B}/\mathcal{O}}^p$ of degree $k+n$ and any $b \in \mathcal{B}$ of degree 1, one has:

$$d\left(\frac{\omega_{k+n}}{b^n}\right) = \frac{b^n d\omega_{k+n} - (db^n) \wedge \omega_{k+n}}{b^{2n}}.$$

Thus, for any $n \in \mathbb{Z}$, one has (\mathcal{O} -linear) morphisms of sheaves

$$d: \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^p(n) \rightarrow \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^{p+1}(n)$$

and we obtain, for each n , a complex of sheaves on \mathbb{P} :

$$\widetilde{\mathrm{DeRham}}(\mathcal{M}, n) = 0 \rightarrow \mathcal{O}_{\mathbb{P}}(n) \xrightarrow{d} \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}(n) \xrightarrow{d} \cdots \xrightarrow{d} \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^p(n) \rightarrow \cdots$$

which can be reformulated as

$$0 \rightarrow \mathcal{O}_{\mathbb{P}}(n) \xrightarrow{d} (\pi^* \mathcal{M})(n-1) \rightarrow \cdots \rightarrow (\pi^* \Lambda^p \mathcal{M})(n-p) \rightarrow \cdots$$

It should be noticed that $\widetilde{\mathrm{DeRham}}(\mathcal{M}, n)$ is not the complex obtained for $n = 0$ twisted by $\mathcal{O}_{\mathbb{P}}(n)$, because the differential is not $\mathcal{O}_{\mathbb{P}}$ -linear.

Again, one has that $i_D \circ d + d \circ i_D = \text{multiplication by } n$, on $\tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^p(n)$. Hence, one has:

Proposition 1.10. *If X is a scheme over \mathbb{Q} , then the complexes $\widetilde{\mathrm{Kos}}(\mathcal{M})(n)$ and $\widetilde{\mathrm{DeRham}}(\mathcal{M}, n)$ are homotopically trivial for any $n \neq 0$.*

Corollary 1.11. *Let X be a scheme over \mathbb{Q} . For any $n \neq 0$, the exact sequences*

$$0 \rightarrow \Omega_{\mathbb{P}/X}^p(n) \rightarrow \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^p(n) \rightarrow \Omega_{\mathbb{P}/X}^{p-1}(n) \rightarrow 0$$

split as sheaves of \mathcal{O} -modules (but not as $\mathcal{O}_{\mathbb{P}}$ -modules).

2. GLOBAL EULER SEQUENCE OF A MODULE AND KOSZUL COMPLEXES

Assume that (X, \mathcal{O}) is a k -scheme, where k is a ring (just for simplicity, one could assume that k is another scheme). Let \mathcal{M} be an \mathcal{O} -module and $\mathcal{B} = S^* \mathcal{M}$ the symmetric algebra over \mathcal{O} . Instead of considering the module of Kähler differentials of \mathcal{B} over \mathcal{O} , we shall now consider the module of Kähler differentials over k , that is, $\Omega_{\mathcal{B}/k}$. As it happened with $\Omega_{\mathcal{B}/\mathcal{O}}$ (section 1), the module $\Omega_{\mathcal{B}/k}$ is a graded \mathcal{B} -module in a natural way. The \mathcal{O} -derivation $D: \mathcal{B} \rightarrow \mathcal{B}$ is in particular a k -derivation, hence it defines a morphism $i_D: \Omega_{\mathcal{B}/k} \rightarrow \mathcal{B}$, which is nothing but the composition of the natural morphism $\Omega_{\mathcal{B}/k} \rightarrow \Omega_{\mathcal{B}/\mathcal{O}}$ with the inner product $i_D: \Omega_{\mathcal{B}/\mathcal{O}} \rightarrow \mathcal{B}$ defined in section 1. Again we obtain a complex of \mathcal{B} -modules $(\Omega_{\mathcal{B}/k}^p, i_D)$ which we denote by $\text{Kos}(\mathcal{M}/k)$:

$$\cdots \longrightarrow \Omega_{\mathcal{B}/k}^p \xrightarrow{i_D} \Omega_{\mathcal{B}/k}^{p-1} \xrightarrow{i_D} \cdots \xrightarrow{i_D} \Omega_{\mathcal{B}/k} \xrightarrow{i_D} \mathcal{B} \longrightarrow 0 \quad (2.1)$$

and for each $n \geq 0$ a complex of \mathcal{O} -modules

$$\text{Kos}(\mathcal{M}/k)_n = \cdots \longrightarrow [\Omega_{\mathcal{B}/k}^p]_n \xrightarrow{i_D} \cdots \longrightarrow [\Omega_{\mathcal{B}/k}]_n \xrightarrow{i_D} S^n \mathcal{M} \longrightarrow 0$$

By homogeneous localization one has a complex of $\mathcal{O}_{\mathbb{P}}$ -modules, denoted by $\widetilde{\text{Kos}}(\mathcal{M}/k)$:

$$\cdots \longrightarrow \widetilde{\Omega}_{\mathcal{B}/k}^p \xrightarrow{i_D} \widetilde{\Omega}_{\mathcal{B}/k}^{p-1} \xrightarrow{i_D} \cdots \longrightarrow \widetilde{\Omega}_{\mathcal{B}/k} \xrightarrow{i_D} \mathcal{O}_{\mathbb{P}} \longrightarrow 0$$

Theorem 2.1. *The complex $\widetilde{\text{Kos}}(\mathcal{M}/k)$ is acyclic (that is, an exact sequence). Moreover,*

$$\Omega_{\mathbb{P}/k}^p = \text{Ker} \left(\widetilde{\Omega}_{\mathcal{B}/k}^p \xrightarrow{i_D} \widetilde{\Omega}_{\mathcal{B}/k}^{p-1} \right).$$

Hence one has exact sequences

$$0 \rightarrow \Omega_{\mathbb{P}/k}^p \rightarrow \widetilde{\Omega}_{\mathcal{B}/k}^p \rightarrow \Omega_{\mathbb{P}/k}^{p-1} \rightarrow 0$$

and right and left resolutions of $\Omega_{\mathbb{P}/k}^p$:

$$\begin{aligned} 0 \rightarrow \Omega_{\mathbb{P}/k}^p \rightarrow \widetilde{\Omega}_{\mathcal{B}/k}^p \rightarrow \widetilde{\Omega}_{\mathcal{B}/k}^{p-1} \rightarrow \cdots \rightarrow \widetilde{\Omega}_{\mathcal{B}/k} \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow 0 \\ \cdots \rightarrow \widetilde{\Omega}_{\mathcal{B}/k}^e \rightarrow \widetilde{\Omega}_{\mathcal{B}/k}^{e-1} \rightarrow \cdots \rightarrow \widetilde{\Omega}_{\mathcal{B}/k}^{p+1} \rightarrow \Omega_{\mathbb{P}/k}^p \rightarrow 0 \end{aligned}$$

In particular, for $p = 1$ the exact sequence

$$0 \rightarrow \Omega_{\mathbb{P}/k} \rightarrow \widetilde{\Omega}_{\mathcal{B}/k} \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow 0 \quad (2.2)$$

is called the (global) Euler sequence.

Proof. It is completely analogous to the proof of Theorem 1.4. \square

Let us denote $\widetilde{\text{Kos}}(\mathcal{M}/k)_n := \pi_*(\widetilde{\text{Kos}}(\mathcal{M}/k)(n))$. The natural morphisms

$$[\Omega_{\mathcal{B}/k}^p]_n \rightarrow \pi_*(\widetilde{\Omega}_{\mathcal{B}/k}^p(n))$$

give a morphism of complexes

$$\text{Kos}(\mathcal{M}/k)_n \rightarrow \widetilde{\text{Kos}}(\mathcal{M}/k)_n.$$

In complete analogy to the relative setting we have the following:

Theorem 2.2. *Let \mathcal{M} be a finitely generated quasi-coherent module on a scheme (X, \mathcal{O}) , $\mathcal{B} = S^* \mathcal{M}$, $\mathbb{P} = \text{Proj } \mathcal{B}$ and $\pi: \mathbb{P} \rightarrow X$ the natural morphism. Let d' be the greatest integer such that $\Omega_{\mathcal{B}/k}^{d'} \neq 0$ and $n > 0$. Then:*

- (1) *If $R^j \pi_* (\widetilde{\Omega}_{\mathcal{B}/k}^i(n)) = 0$ for any $j > 0$ and any $0 \leq i \leq d'$, then $\widetilde{\text{Kos}}(\mathcal{M}/k)_n$ is acyclic.*
- (2) *If (1) holds and the natural morphism $[\Omega_{\mathcal{B}/k}^i]_n \rightarrow \pi_* (\widetilde{\Omega}_{\mathcal{B}/k}^i(n))$ is an isomorphism for any $0 \leq i \leq d'$, then $\text{Kos}(\mathcal{M}/k)_n$ is also acyclic.*

Theorem 2.3. *Let X be a noetherian scheme and \mathcal{M} a coherent module on X . The Koszul complexes $\text{Kos}(\mathcal{M}/k)_n$ and $\widetilde{\text{Kos}}(\mathcal{M}/k)_n$ are acyclic for $n > 0$.*

2.1. Koszul versus De Rham (Global case). Now we pass to the De Rham complex (over k). The k -linear differentials

$$d: \Omega_{\mathcal{B}/k}^p \rightarrow \Omega_{\mathcal{B}/k}^{p+1}$$

give a (global) De Rham complex

$$\text{DeRham}(\mathcal{M}/k) \equiv 0 \rightarrow \mathcal{B} \xrightarrow{d} \Omega_{\mathcal{B}/k} \xrightarrow{d} \cdots \xrightarrow{d} \Omega_{\mathcal{B}/k}^{p-1} \xrightarrow{d} \Omega_{\mathcal{B}/k}^p \rightarrow \cdots$$

which is bounded if X is of finite type over k . Since d is homogeneous of degree 0, one has for each $n \geq 0$ a complex of \mathcal{O} -modules (with k -linear differential)

$$\text{DeRham}(\mathcal{M}/k)_n \equiv 0 \rightarrow S^n \mathcal{M} \xrightarrow{d} [\Omega_{\mathcal{B}/k}]_n \xrightarrow{d} \cdots \xrightarrow{d} [\Omega_{\mathcal{B}/k}^p]_n \rightarrow \cdots$$

One has again Cartan's formula: $i_D \circ d + d \circ i_D = \text{multiplication by } n$, on $[\Omega_{\mathcal{B}/k}^p]_n$ and then:

Proposition 2.4. *If X is a scheme over \mathbb{Q} , then $\text{Kos}(\mathcal{M}/k)_n$ and $\text{DeRham}(\mathcal{M}/k)_n$ are homotopically trivial (in particular, acyclic) for any $n > 0$.*

As in section 1.2, we can take homogeneous localizations: for each $n \in \mathbb{Z}$, the differentials $\Omega_{\mathcal{B}/k}^p \rightarrow \Omega_{\mathcal{B}/k}^{p+1}$ induce k -linear morphisms

$$d: \widetilde{\Omega}_{\mathcal{B}/k}^p(n) \rightarrow \widetilde{\Omega}_{\mathcal{B}/k}^{p+1}(n)$$

and one obtains a complex of $\mathcal{O}_{\mathbb{P}}$ -modules (with k -linear differential)

$$\widetilde{\text{DeRham}}(\mathcal{M}/k, n) = 0 \rightarrow \mathcal{O}_{\mathbb{P}}(n) \xrightarrow{d} \widetilde{\Omega}_{\mathcal{B}/k}(n) \xrightarrow{d} \cdots \xrightarrow{d} \widetilde{\Omega}_{\mathcal{B}/k}^p(n) \rightarrow \cdots$$

Again, the differentials of Koszul and De Rham complexes are related by Cartan's formula: $i_D \circ d + d \circ i_D = \text{multiplication by } n$, on $\widetilde{\Omega}_{\mathcal{B}/k}^p(n)$, so one has:

Proposition 2.5. *Let X be a scheme over \mathbb{Q} . The complexes $\widetilde{\text{Kos}}(\mathcal{M}/k)(n)$ and $\widetilde{\text{DeRham}}(\mathcal{M}/k, n)$ are homotopically trivial (in particular, acyclic) for any $n \neq 0$.*

Corollary 2.6. *If X is a scheme over \mathbb{Q} , then for any $n \neq 0$, the exact sequences*

$$0 \rightarrow \Omega_{\mathbb{P}/k}^p(n) \rightarrow \widetilde{\Omega}_{\mathcal{B}/k}^p(n) \rightarrow \Omega_{\mathbb{P}/k}^{p-1}(n) \rightarrow 0$$

split as sheaves of k -modules (but not as $\mathcal{O}_{\mathbb{P}}$ -modules).

3. COHOMOLOGY OF PROJECTIVE BUNDLES

In this section we assume that \mathcal{E} is a locally free sheaf of rank $r+1$ on a k -scheme (X, \mathcal{O}) . Let $\mathcal{B} = S^*\mathcal{E}$ be its symmetric algebra over \mathcal{O} and $\mathbb{P} = \text{Proj } \mathcal{B} \xrightarrow{\pi} X$ the corresponding projective bundle. Our aim is to determine the cohomology of the sheaves $\Omega_{\mathbb{P}/X}^p(n)$ and $\Omega_{\mathbb{P}/k}^p(n)$.

3.1. Cohomology of $\Omega_{\mathbb{P}/X}^p(n)$.

Notations: In order to simplify some statements, we shall use the following conventions:

- (1) $S^p\mathcal{E} = 0$ whenever $p < 0$, and analogously for exterior powers.
- (2) For any integer p , we shall denote $\bar{p} = r + 1 - p$.
- (3) For any \mathcal{O} -module \mathcal{M} , we shall denote by \mathcal{M}^* its dual: $\mathcal{M}^* = \text{Hom}(\mathcal{M}, \mathcal{O})$.

We shall use the following well known result on the cohomology of a projective bundle:

Proposition 3.1. *Let n be a non negative integer. Then*

$$R^i\pi_*\mathcal{O}_{\mathbb{P}}(n) = \begin{cases} 0 & \text{for } i \neq 0 \\ S^n\mathcal{E} & \text{for } i = 0 \end{cases}$$

If n is a positive integer, then

$$R^i\pi_*\mathcal{O}_{\mathbb{P}}(-n) = \begin{cases} 0 & \text{for } i \neq r \\ S^{n-r-1}\mathcal{E}^* \otimes \Lambda^{r+1}\mathcal{E}^* & \text{for } i = r \end{cases}$$

We shall also use without further explanation a particular case of projection formula: for any quasi-coherent module \mathcal{N} on X and any locally free module \mathcal{L} on \mathbb{P} such that $R^j\pi_*\mathcal{L}$ is locally free (for any j), one has

$$R^i\pi_*(\pi^*\mathcal{N} \otimes \mathcal{L}) = \mathcal{N} \otimes R^i\pi_*\mathcal{L}.$$

Proposition 3.2. *Let n be a non negative integer. Then*

$$R^i\pi_*\tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^p(n) = \begin{cases} 0 & \text{for } i \neq 0 \\ \Lambda^p\mathcal{E} \otimes S^{n-p}\mathcal{E} & \text{for } i = 0 \end{cases}$$

For any positive integer n , one has

$$R^i\pi_*\tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^p(-n) = \begin{cases} 0 & \text{for } i \neq r \\ \Lambda^{\bar{p}}\mathcal{E}^* \otimes S^{n-\bar{p}}\mathcal{E}^* & \text{for } i = r \end{cases} \quad \text{with } \bar{p} = r + 1 - p$$

Proof. Since $\tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^p = (\pi^*\Lambda^p\mathcal{E})(-p)$, the results follows from Proposition 3.1. For the second formula we have also used the natural isomorphism $\Lambda^{\bar{p}}\mathcal{E} = \Lambda^p\mathcal{E}^* \otimes \Lambda^{r+1}\mathcal{E}$. \square

Remark 3.3. Notice that $\Lambda^p\mathcal{E} \otimes S^{n-p}\mathcal{E} = [\Omega_{\mathcal{B}/\mathcal{O}}^p]_n$. Thus, Proposition 3.2 and Theorem 1.5 tell us that $\text{Kos}(\mathcal{E})_n \rightarrow \widetilde{\text{Kos}}(\mathcal{E})_n$ is an isomorphism for any $n \geq 0$ and the Koszul complexes $\widetilde{\text{Kos}}(\mathcal{E})_n$ and $\text{Kos}(\mathcal{E})_n$ are acyclic for any $n > 0$ (thus we obtain the well known fact of the acyclicity of the Koszul complex of a locally free module).

Let us denote by $\mathcal{K}_{p,n}$ the kernels of the morphisms i_D in $\text{Kos}(\mathcal{E})_n$, that is,

$$\mathcal{K}_{p,n} := \text{Ker}(\Lambda^p\mathcal{E} \otimes S^{n-p}\mathcal{E} \rightarrow \Lambda^{p-1}\mathcal{E} \otimes S^{n-p+1}\mathcal{E})$$

One has the following result (see [12] or [4, Exposé XI] for different approaches).

Theorem 3.4. *Let \mathcal{E} be a locally free sheaf of rank $r + 1$ on a k -scheme (X, \mathcal{O}) and $\mathbb{P} = \text{Proj } S\mathcal{E} \xrightarrow{\pi} X$ the corresponding projective bundle.*

Let n be a positive integer number.

(1)

$$R^i \pi_* \Omega_{\mathbb{P}/X}^p = \begin{cases} \mathcal{O} & \text{if } 0 \leq i = p \leq r \\ 0 & \text{otherwise} \end{cases}.$$

(2)

$$R^i \pi_* \Omega_{\mathbb{P}/X}^p(n) = \begin{cases} 0 & \text{if } i \neq 0 \\ \mathcal{K}_{p,n} & \text{if } i = 0 \end{cases}$$

and, if X is a \mathbb{Q} -scheme, then

$$\mathcal{K}_{p,n} \oplus \mathcal{K}_{p-1,n} = \Lambda^p \mathcal{E} \otimes S^{n-p} \mathcal{E}.$$

(3)

$$R^i \pi_* \Omega_{\mathbb{P}/X}^p(-n) = \begin{cases} 0 & \text{if } i \neq r \\ \mathcal{K}_{r-p,n}^* & \text{if } i = r \end{cases}$$

and, if X is a \mathbb{Q} -scheme, then

$$\mathcal{K}_{r-p,n}^* \oplus \mathcal{K}_{r-p+1,n}^* = \Lambda^{\bar{p}} \mathcal{E}^* \otimes S^{n-\bar{p}} \mathcal{E}^*.$$

Proof. Let $n \geq 0$. By Theorem 1.4

$$0 \rightarrow \Omega_{\mathbb{P}/X}^p(n) \rightarrow \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^p(n) \rightarrow \cdots \rightarrow \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}(n) \rightarrow \mathcal{O}_{\mathbb{P}}(n) \rightarrow 0$$

is a resolution of $\Omega_{\mathbb{P}/X}^p(n)$ by π_* -acyclic sheaves (by Proposition 3.2). One concludes then by Proposition 3.2 and Remark 3.3.

(3) follows from (2) and (relative) Grothendieck duality: one has an isomorphism $\Omega_{\mathbb{P}/X}^p = \mathcal{H}om(\Omega_{\mathbb{P}/X}^{r-p}, \Omega_{\mathbb{P}/X}^r)$ and then

$$\mathbb{R}\pi_* \Omega_{\mathbb{P}/X}^p(-n) \simeq \mathbb{R}\pi_* \mathcal{H}om(\Omega_{\mathbb{P}/X}^{r-p}(n), \Omega_{\mathbb{P}/X}^r) \simeq \mathbb{R}\mathcal{H}om(\mathbb{R}\pi_* \Omega_{\mathbb{P}/X}^{r-p}(n)[r], \mathcal{O})$$

and one concludes by (2).

Finally, the statements of (2) and (3) regarding the case that X is a \mathbb{Q} -scheme follow from Corollary 1.11. \square

Corollary 3.5. *(Bott's formula) Let \mathbb{P}_r be the projective space of dimension r over a field k . Let n be a positive integer number.*

(1)

$$\dim_k H^q(\mathbb{P}_r, \Omega_{\mathbb{P}_r}^p) = \begin{cases} 1 & \text{if } 0 \leq q = p \leq r \\ 0 & \text{otherwise} \end{cases}$$

(2)

$$\dim_k H^q(\mathbb{P}_r, \Omega_{\mathbb{P}_r}^p(n)) = \begin{cases} 0 & \text{if } q \neq 0 \\ \binom{n+r-p}{n} \binom{n-1}{p} & \text{if } q = 0 \end{cases}$$

(3)

$$\dim_k H^q(\mathbb{P}_r, \Omega_{\mathbb{P}_r}^p(-n)) = \begin{cases} 0 & \text{if } q \neq r \\ \binom{n+p}{n} \binom{n-1}{r-p} & \text{if } q = r \end{cases}$$

Proof. It follows from Theorem 3.4, once one proves that $\dim_k \mathcal{K}_{p,n} = \binom{n+r-p}{n} \binom{n-1}{p}$. From the exact sequence $0 \rightarrow \mathcal{K}_{p,n} \rightarrow \Lambda^p \mathcal{E} \otimes S^{n-p} \mathcal{E} \rightarrow \mathcal{K}_{p-1,n} \rightarrow 0$ it follows that $\dim_k \mathcal{K}_{p,n} + \dim_k \mathcal{K}_{p-1,n} = \binom{r+1}{p} \binom{n-p+r}{r}$; hence it suffices to prove that

$$\binom{n+r-p}{n} \binom{n-1}{p} + \binom{n+r-p+1}{n} \binom{n-1}{p-1} = \binom{r+1}{p} \binom{n-p+r}{r}$$

which is an easy computation if one writes $\binom{a}{b} = \frac{a!}{b!(a-b)!}$. \square

Remark 3.6. (1) We can give an interpretation of $H^0(\mathbb{P}_r, \Omega_{\mathbb{P}_r}^p(n))$ in terms of differential forms of the polynomial ring $k[x_0, \dots, x_r]$; one has the exact sequence

$$0 \rightarrow H^0(\mathbb{P}_r, \Omega_{\mathbb{P}_r}^p(n)) \rightarrow [\Omega_{k[x_0, \dots, x_r]/k}^p]_n \xrightarrow{i_D} [\Omega_{k[x_0, \dots, x_r]/k}^{p-1}]_n$$

that is, $H^0(\mathbb{P}_r, \Omega_{\mathbb{P}_r}^p(n))$ are those p -forms $\omega_p \in \Omega_{k[x_0, \dots, x_r]/k}^p$ which are homogeneous of degree n and such that $i_D \omega_p = 0$, where $D = \sum_{i=0}^r x_i \frac{\partial}{\partial x_i}$.

(2) From the exact sequence

$$0 \rightarrow H^0(\mathbb{P}_r, \Omega_{\mathbb{P}_r}^p(n)) \rightarrow \Lambda^p \mathcal{E} \otimes S^{n-p} \mathcal{E} \rightarrow \dots \rightarrow \mathcal{E} \otimes S^{n-1} \mathcal{E} \rightarrow S^n \mathcal{E} \rightarrow 0$$

we can give a different combinatorial expression of $\dim_k H^0(\mathbb{P}_r, \Omega_{\mathbb{P}_r}^p(n))$ (as Verdier does):

$$\dim_k H^0(\mathbb{P}_r, \Omega_{\mathbb{P}_r}^p(n)) = \sum_{i=0}^p (-1)^i \binom{r+1}{p-i} \binom{n+r-p+i}{r}.$$

It follows from Theorem 3.4 that $H^q(\mathbb{P}, \Omega_{\mathbb{P}/X}^p) = H^{q-p}(X, \mathcal{O})$. For the twisted case we have the following:

Corollary 3.7. *Let X be a proper scheme over a field k of characteristic zero. Let \mathcal{E} be a locally free module on X of rank $r+1$ and $\mathbb{P} = \text{Proj } S^* \mathcal{E}$ the associated projective bundle. Then, for any positive integer n , one has:*

- (1) $\dim_k H^q(\mathbb{P}, \Omega_{\mathbb{P}/X}^p(n)) = \sum_{i=0}^p (-1)^i \dim H^q(X, \Lambda^{p-i} \mathcal{E} \otimes S^{n-p+i} \mathcal{E})$.
- (2) $\dim_k H^q(\mathbb{P}, \Omega_{\mathbb{P}/X}^p(-n)) = \sum_{i=0}^p (-1)^i \dim H^{q-r}(X, \Lambda^{\bar{p}+i} \mathcal{E}^* \otimes S^{n-\bar{p}-i} \mathcal{E}^*)$
with $\bar{p} = r+1-p$.

Proof. (1) By Corollary 1.11, one has

$$H^q(\mathbb{P}, \Omega_{\mathbb{P}/X}^p(n)) \oplus H^q(\mathbb{P}, \Omega_{\mathbb{P}/X}^{p-1}(n)) = H^q(\mathbb{P}, \tilde{\Omega}_{\mathbb{B}/\mathcal{O}}^p(n))$$

and $H^q(\mathbb{P}, \tilde{\Omega}_{\mathbb{B}/\mathcal{O}}^p(n)) = H^q(X, \Lambda^p \mathcal{E} \otimes S^{n-p} \mathcal{E})$ by Proposition 3.2. Conclusion follows.

(2) is completely analogous. \square

3.2. Cohomology of $\Omega_{\mathbb{P}/k}^p(n)$.

Let us consider the exact sequence of differentials

$$0 \rightarrow \Omega_{X/k} \otimes_{\mathcal{O}} \mathcal{B} \rightarrow \Omega_{\mathcal{B}/k} \rightarrow \Omega_{\mathcal{B}/\mathcal{O}} \rightarrow 0$$

This sequence locally splits: indeed, if \mathcal{E} is trivial, then $\mathcal{E} = E \otimes_k \mathcal{O}$ and $\mathcal{B} = B \otimes_k \mathcal{O}$, with $B = S^* E$; hence, $\Omega_{\mathcal{B}/\mathcal{O}} = \Omega_{B/k} \otimes_k \mathcal{O}$ and there is a natural morphism $\Omega_{B/k} \otimes_k \mathcal{O} \rightarrow \Omega_{\mathcal{B}/k}$ which is a section of $\Omega_{\mathcal{B}/k} \rightarrow \Omega_{\mathcal{B}/\mathcal{O}}$.

Remark 3.8. The exact sequence is a sequence of graded \mathcal{B} -modules, hence it gives an exact sequence of \mathcal{O} -modules in each degree. In particular, in degree 0 one obtains an isomorphism $\Omega_{X/k} = [\Omega_{\mathcal{B}/k}]_0$, and an exact sequence in degree 1:

$$0 \rightarrow \Omega_{X/k} \otimes_{\mathcal{O}} \mathcal{E} \rightarrow [\Omega_{\mathcal{B}/k}]_1 \rightarrow \mathcal{E} \rightarrow 0$$

which is nothing but the Atiyah extension.

Taking homogeneous localizations we obtain an exact sequence of $\mathcal{O}_{\mathbb{P}}$ -modules

$$0 \rightarrow \pi^* \Omega_{X/k} \rightarrow \tilde{\Omega}_{\mathcal{B}/k} \rightarrow \tilde{\Omega}_{\mathcal{B}/\mathcal{O}} \rightarrow 0$$

which splits locally (on X).

Proposition 3.9. *Let n be a positive integer. Then:*

(1)

$$R^i \pi_* \tilde{\Omega}_{\mathcal{B}/k}^p = \begin{cases} 0 & \text{for } i \neq 0, r \\ \Omega_{X/k}^p & \text{for } i = 0 \\ \Omega_{X/k}^{p-r-1} & \text{for } i = r \end{cases}.$$

(2)

$$R^i \pi_* \tilde{\Omega}_{\mathcal{B}/k}^p(n) = \begin{cases} 0 & \text{for } i \neq 0 \\ [\Omega_{\mathcal{B}/k}^p]_n & \text{for } i = 0 \end{cases}$$

(3) $R^i \pi_* \tilde{\Omega}_{\mathcal{B}/k}^p(-n) = 0$ for $i \neq r$ and $R^r \pi_* \tilde{\Omega}_{\mathcal{B}/k}^p(-n)$ is locally isomorphic to $\bigoplus_{q=0}^p (\Omega_{X/k}^{p-q} \otimes \Lambda^{\bar{q}} \mathcal{E}^* \otimes S^{n-\bar{q}} \mathcal{E}^*)$, with $\bar{q} = r + 1 - q$.

(4) Furthermore, if X is a smooth k -scheme (of relative dimension d), then

$$R^r \pi_* \tilde{\Omega}_{\mathcal{B}/k}^p(-n) = [\Omega_{\mathcal{B}/k}^{d+p}]_n^* \otimes \Omega_{X/k}^d$$

Proof. If \mathcal{E} is trivial, then $\tilde{\Omega}_{\mathcal{B}/k} = \pi^* \Omega_{X/k} \oplus \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}$, so $\tilde{\Omega}_{\mathcal{B}/k}^p = \bigoplus_{q=0}^p \pi^* \Omega_{X/k}^{p-q} \otimes \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^q$ and (1)-(3) follow from Proposition 3.2 in this case. Since \mathcal{E} is locally trivial, we obtain the vanishing statements of (1)-(3).

(1) The natural morphism $\Omega_{X/k}^p \rightarrow \pi_* \tilde{\Omega}_{\mathcal{B}/k}^p$ is an isomorphism because it is locally so. The natural morphism $\tilde{\Omega}_{\mathcal{B}/k}^{r+1} \rightarrow \Omega_{\mathcal{B}/\mathcal{O}}^{r+1}$ gives a morphism $R^r \pi_* \tilde{\Omega}_{\mathcal{B}/k}^{r+1} \rightarrow R^r \pi_* \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^{r+1} = \mathcal{O}$, which is an isomorphism because it is locally so. Finally, for any $p \geq 0$, the natural morphism $\tilde{\Omega}_{\mathcal{B}/k}^p \otimes \tilde{\Omega}_{\mathcal{B}/k}^{r+1} \rightarrow \tilde{\Omega}_{\mathcal{B}/k}^{p+r+1}$ induces a morphism $\pi_*(\tilde{\Omega}_{\mathcal{B}/k}^p) \otimes R^r \pi_*(\tilde{\Omega}_{\mathcal{B}/k}^{r+1}) \rightarrow R^r \pi_* \tilde{\Omega}_{\mathcal{B}/k}^{p+r+1}$, i.e. a morphism $\Omega_{X/k}^p \rightarrow R^r \pi_* \tilde{\Omega}_{\mathcal{B}/k}^{p+r+1}$, which is an isomorphism because it is locally so.

(2) The natural morphism $[\Omega_{\mathcal{B}/k}^p]_n \rightarrow \pi_* \tilde{\Omega}_{\mathcal{B}/k}^p(n)$ is an isomorphism because it is locally so.

It only remains to prove (4), which is a consequence of (relative) Grothendieck duality. Indeed, notice that, under the smoothness hypothesis, $R^r \pi_* \tilde{\Omega}_{\mathcal{B}/k}^p(-n)$ is locally free, by (3). Hence, it suffices to compute its dual. This is given by duality: the relative dualizing sheaf is $\Omega_{\mathbb{P}/X}^r = \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^{r+1}$ and one has isomorphisms

$\tilde{\Omega}_{\mathcal{B}/k}^{d+r+1} = \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^{r+1} \otimes \pi^* \Omega_{X/k}^d$ and $\mathcal{H}om(\tilde{\Omega}_{\mathcal{B}/k}^p, \tilde{\Omega}_{\mathcal{B}/k}^{d+r+1}) = \tilde{\Omega}_{\mathcal{B}/k}^{d+\bar{p}}$; then:

$$\begin{aligned} \left[R^r \pi_* \tilde{\Omega}_{\mathcal{B}/k}^p(-n) \right]^* &= \pi_* \mathcal{H}om_{\mathbb{P}}(\tilde{\Omega}_{\mathcal{B}/k}^p(-n), \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^{r+1}) \\ &= \pi_* [\mathcal{H}om_{\mathbb{P}}(\tilde{\Omega}_{\mathcal{B}/k}^p(-n), \tilde{\Omega}_{\mathcal{B}/k}^{d+r+1}) \otimes \pi^*(\Omega_{X/k}^d)^*] \\ &= (\pi_* \tilde{\Omega}_{\mathcal{B}/k}^{d+\bar{p}}(n)) \otimes (\Omega_{X/k}^d)^* \stackrel{(2)}{=} [\Omega_{\mathcal{B}/k}^{d+\bar{p}}]_n \otimes (\Omega_{X/k}^d)^*. \end{aligned}$$

□

Corollary 3.10. *The Koszul complexes $\text{Kos}(\mathcal{E}/k)_n$ and $\widetilde{\text{Kos}}(\mathcal{E}/k)_n$ are acyclic for $n > 0$ and $\text{Kos}(\mathcal{E}/k)_n \rightarrow \widetilde{\text{Kos}}(\mathcal{E}/k)_n$ is an isomorphism for any $n \geq 0$.*

Let us denote by $\overline{\mathcal{K}}_{p,n}$ the kernels of the morphisms i_D in the Koszul complex $\text{Kos}(\mathcal{E}/k)_n$; that is,

$$\overline{\mathcal{K}}_{p,n} := \text{Ker} \left([\Omega_{\mathcal{B}/k}^p]_n \rightarrow [\Omega_{\mathcal{B}/k}^{p-1}]_n \right)$$

Theorem 3.11. *Let \mathcal{E} be a locally free sheaf of rank $r+1$ on a k -scheme (X, \mathcal{O}) and $\mathbb{P} = \text{Proj } S \cdot \mathcal{E} \xrightarrow{\pi} X$ the corresponding projective bundle.*

Let n be a positive integer. One has:

- (1) $R^i \pi_* \Omega_{\mathbb{P}/k}^p = \Omega_{X/k}^{p-i}$.
- (2)

$$R^i \pi_* \Omega_{\mathbb{P}/k}^p(n) = \begin{cases} 0 & \text{for } i \neq 0 \\ \overline{\mathcal{K}}_{p,n} & \text{for } i = 0 \end{cases}$$

and, if X is a \mathbb{Q} -scheme, then one has an isomorphism (of k -modules, not of \mathcal{O} -modules)

$$\overline{\mathcal{K}}_{p,n} \oplus \overline{\mathcal{K}}_{p-1,n} = [\Omega_{\mathcal{B}/k}^p]_n.$$

- (3) $R^i \pi_* \Omega_{\mathbb{P}/k}^p(-n) = 0$ for $i \neq r$ and $R^r \pi_* \Omega_{\mathbb{P}/k}^p(-n)$ is locally isomorphic to $\bigoplus_{q=0}^p \Omega_{X/k}^{p-q} \otimes \mathcal{K}_{r-q,n}^*$. Moreover, if X is a \mathbb{Q} -scheme, then one has an isomorphism (of k -modules, not of \mathcal{O} -modules)

$$R^r \pi_* \Omega_{\mathbb{P}/k}^p(-n) \oplus R^r \pi_* \Omega_{\mathbb{P}/k}^{p-1}(-n) = R^r \pi_* \tilde{\Omega}_{\mathcal{B}/k}^p(-n)$$

- (4) *If X is a smooth k -scheme (of relative dimension d), then*

$$R^r \pi_* \Omega_{\mathbb{P}/k}^p(-n) = \overline{\mathcal{K}}_{d+r-p,n}^* \otimes \Omega_{X/k}^d$$

and, if X is a \mathbb{Q} -scheme, then one has an isomorphism (of k -modules, not of \mathcal{O} -modules)

$$R^r \pi_* \Omega_{\mathbb{P}/k}^p(-n) \oplus R^r \pi_* \Omega_{\mathbb{P}/k}^{p-1}(-n) = \left[\Omega_{\mathcal{B}/k}^{d+\bar{p}} \right]_n^* \otimes \Omega_{X/k}^d.$$

Proof. If \mathcal{E} is trivial, then $\Omega_{\mathbb{P}/k} = \pi^* \Omega_{X/k} \oplus \Omega_{\mathbb{P}/X}$, so $\Omega_{\mathbb{P}/k}^p = \bigoplus_{q=0}^p \pi^* \Omega_{X/k}^q \otimes \Omega_{\mathbb{P}/X}^{p-q}$ and (1)-(3) follow from Theorem 3.4 in this case. Since \mathcal{E} is locally trivial, we obtain the vanishing statements of (1)-(3).

- (1) The exact sequences $0 \rightarrow \Omega_{\mathbb{P}/k}^p \rightarrow \tilde{\Omega}_{\mathcal{B}/k}^p \rightarrow \Omega_{\mathbb{P}/k}^{p-1} \rightarrow 0$ induce morphisms

$$\pi_* \Omega_{\mathbb{P}/k}^{p-i} \rightarrow R^1 \pi_* \Omega_{\mathbb{P}/k}^{p-i+1} \rightarrow \cdots \rightarrow R^i \pi_* \Omega_{\mathbb{P}/k}^p$$

whose composition with the natural morphism $\Omega_{X/k}^{p-i} \rightarrow \pi_* \Omega_{\mathbb{P}/k}^{p-i}$ gives a morphism $\Omega_{X/k}^{p-i} \rightarrow R^i \pi_* \Omega_{\mathbb{P}/k}^p$. This morphism is an isomorphism because it is locally so.

(2) The exact sequence $0 \rightarrow \Omega_{\mathbb{P}/k}^p(n) \rightarrow \tilde{\Omega}_{\mathcal{B}/k}^p(n) \rightarrow \tilde{\Omega}_{\mathcal{B}/k}^{p-1}(n)$ induces, taking direct image, the isomorphism $\pi_* \Omega_{\mathbb{P}/k}^p(n) = \overline{\mathcal{K}}_{p,n}$.

(4) follows from (2) and (relative) Grothendieck duality. Indeed, notice that, under the smoothness hypothesis, $R^r \pi_* \Omega_{\mathbb{P}/k}^p(-n)$ is locally free, by (3). Hence, it suffices to compute its dual. This is given by duality: the relative dualizing sheaf is $\Omega_{\mathbb{P}/X}^r$ and one has isomorphisms $\Omega_{\mathbb{P}/k}^{d+r} = \Omega_{\mathbb{P}/X}^r \otimes \pi^* \Omega_{X/k}^d$ and $\mathcal{H}om(\Omega_{\mathbb{P}/k}^p, \Omega_{\mathbb{P}/k}^{d+r}) = \Omega_{\mathbb{P}/k}^{d+r-p}$; then:

$$\begin{aligned} \left[R^r \pi_* \Omega_{\mathbb{P}/k}^p(-n) \right]^* &= \pi_* \mathcal{H}om_{\mathbb{P}}(\Omega_{\mathbb{P}/k}^p(-n), \Omega_{\mathbb{P}/k}^r) \\ &= \pi_* [\mathcal{H}om_{\mathbb{P}}(\Omega_{\mathbb{P}/k}^p(-n), \Omega_{\mathbb{P}/k}^{d+r}) \otimes \pi^*(\Omega_{X/k}^d)^*] \\ &= (\pi_* \tilde{\Omega}_{\mathbb{P}/k}^{d+r-p}(n)) \otimes (\Omega_{X/k}^d)^* \stackrel{(2)}{=} \overline{\mathcal{K}}_{d+r-p,n} \otimes (\Omega_{X/k}^d)^*. \end{aligned}$$

Finally, the statements of (2)-(4) regarding the case of a \mathbb{Q} -scheme follow from Corollary 2.6. \square

Remark 3.12. For $n = 1$ a little more can be said (as Verdier does): The natural morphism $\Omega_{X/k}^p \otimes \mathcal{E} \rightarrow \pi_* \Omega_{\mathbb{P}/k}^p(1)$ is an isomorphism. Indeed, the exact sequence

$$0 \rightarrow \Omega_{X/k} \otimes \mathcal{B} \rightarrow \Omega_{\mathcal{B}/k} \rightarrow \Omega_{\mathcal{B}/\mathcal{O}} \rightarrow 0$$

induces for each p an exact sequence

$$0 \rightarrow \Omega_{X/k}^p \otimes \mathcal{B} \rightarrow \Omega_{\mathcal{B}/k}^p \rightarrow \Omega_{\mathcal{B}/k}^{p-1} \otimes \Omega_{\mathcal{B}/\mathcal{O}} \rightarrow \Omega_{\mathcal{B}/k}^{p-2} \otimes S^2 \Omega_{\mathcal{B}/\mathcal{O}} \rightarrow \cdots$$

and taking degree 1, an exact sequence

$$0 \rightarrow \Omega_{X/k}^p \otimes \mathcal{E} \rightarrow [\Omega_{\mathcal{B}/k}^p]_1 \rightarrow \Omega_{X/k}^{p-1} \otimes \mathcal{E} \rightarrow 0$$

On the other hand, taking π_* in the exact sequence

$$0 \rightarrow \Omega_{\mathbb{P}/k}^p(1) \rightarrow \tilde{\Omega}_{\mathcal{B}/k}^p(1) \rightarrow \Omega_{\mathbb{P}/k}^{p-1}(1) \rightarrow 0$$

gives the exact sequence

$$0 \rightarrow \pi_* \Omega_{\mathbb{P}/k}^p(1) \rightarrow [\Omega_{\mathcal{B}/k}^p]_1 \rightarrow \pi_* \Omega_{\mathbb{P}/k}^{p-1}(1) \rightarrow 0.$$

Thus, the isomorphism $\Omega_{X/k}^p \otimes \mathcal{E} \rightarrow \pi_* \Omega_{\mathbb{P}/k}^p(1)$ is proved by induction on p .

Remark 3.13. It is known (see [1] or [6]) that $\mathbb{R}\pi_* \Omega_{\mathbb{P}/k}^p$ is decomposable, i.e., one has an isomorphism in the derived category $\mathbb{R}\pi_* \Omega_{\mathbb{P}/k}^p = \bigoplus_{i=0}^r \Omega_{X/k}^{p-i}[-i]$. Let us see that, for $p \in [0, r]$, this is a consequence of Theorem 2.1 and Proposition 3.9. Indeed, by Theorem 2.1, one has the exact sequence

$$0 \rightarrow \Omega_{\mathbb{P}/k}^p \rightarrow \tilde{\Omega}_{\mathcal{B}/k}^p \rightarrow \tilde{\Omega}_{\mathcal{B}/k}^{p-1} \rightarrow \cdots \rightarrow \tilde{\Omega}_{\mathcal{B}/k} \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow 0$$

and, by Proposition 3.9, $\tilde{\Omega}_{\mathcal{B}/k}^{p-i}$ are π_* -acyclic for any $i \geq 0$ and $\pi_* \tilde{\Omega}_{\mathcal{B}/k}^{p-i} = \Omega_{X/k}^{p-i}$. Then

$$\mathbb{R}\pi_* \Omega_{\mathbb{P}/k}^p \equiv 0 \rightarrow \Omega_{X/k}^p \rightarrow \Omega_{X/k}^{p-1} \rightarrow \cdots \rightarrow \Omega_{X/k} \rightarrow \mathcal{O} \rightarrow 0$$

and, since the differential $i_D: \Omega_{X/k}^j \rightarrow \Omega_{X/k}^{j-1}$ is null, we obtain the result.

The decomposability of $\mathbb{R}\pi_*\Omega_{\mathbb{P}/k}^p$ implies an isomorphism

$$H^q(\mathbb{P}, \Omega_{\mathbb{P}/k}^p) = \bigoplus_{i=0}^r H^{q-i}(X, \Omega_{X/k}^{p-i})$$

For the twisted case we have the following:

Corollary 3.14. *Let X be a proper scheme over a field k of characteristic zero. Let \mathcal{E} be a locally free module on X of rank $r+1$ and $\mathbb{P} = \text{Proj } S^*\mathcal{E}$ the associated projective bundle. Then, for any positive integer n , one has:*

- (1) $\dim_k H^q(\mathbb{P}, \Omega_{\mathbb{P}/k}^p(n)) = \sum_{i=0}^p (-1)^i \dim_k H^q(X, [\Omega_{\mathcal{B}/k}^{p-i}]_n)$.
- (2) *If X is smooth over k of dimension d , then*

$$\dim_k H^q(\mathbb{P}, \Omega_{\mathbb{P}/k}^p(-n)) = \sum_{i=0}^{d+r-p} (-1)^i \dim_k H^{d+r-q}(X, [\Omega_{\mathcal{B}/k}^{d+r-p-i}]_n).$$

Proof. (1) By Corollary 2.6,

$$H^q(\mathbb{P}, \Omega_{\mathbb{P}/k}^p(n)) \oplus H^q(\mathbb{P}, \Omega_{\mathbb{P}/k}^{p-1}(n)) = H^q(\mathbb{P}, \tilde{\Omega}_{\mathcal{B}/k}^p(n))$$

and $H^q(\mathbb{P}, \tilde{\Omega}_{\mathcal{B}/k}^p(n)) = H^q(X, [\Omega_{\mathcal{B}/k}^p]_n)$ by Proposition 3.9. Conclusion follows.

(2) follows from (1) and duality. □

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